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Hydrodynamic interaction between gas bubbles in liquid

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A calculation is given of the velocity which a cloud of identical gas bubbles acquires when the liquid in which the cloud is immersed is impulsively accelerated. From the results an expression follows for the effective virtual mass of a bubble in a gas-bubble/liquid mixture. Further consideration is given to that part of the momentum flux in the mixture associated with relative motion between liquid and bubbles. An expression for this quantity is derived which appears to differ from the one used in practice. It is shown that qualitative support for the expression obtained here is provided by experimental observations reported in the literature.

1. Introduction

When a body moves through an unbounded perfect fluid, the momentum of the fluid is indeterminate. A useful quantity, however, is the impulse, which is equal to the relative velocity between fluid and body multiplied by the virtual mass of the body. Kelvin (see Lamb 1932, chap. 6) showed that in an unbounded fluid the external force on the body equals the rate of change of the sum of the impulse of the fluid and the momentum of the body. The dynamics of the relative motion are governed by this relation. In the case of a spherical gas bubble moving through a liquid, the momentum of the bubble is negligible compared with the impulse of the liquid. The virtual mass of a sphere is $\frac{1}{2}\rho\tau$, where ρ is the density of the liquid and τ the volume of the sphere. If the liquid is linearly accelerated with acceleration du/dt, the velocity v of the bubble then follows, according to Kelvin's results, from the relation

$$\frac{d}{dt}\tfrac{1}{2}\rho\tau(v-u) = \rho\tau\frac{du}{dt}. \tag{1.1}$$

The left-hand side is the rate of change of the impulse of the liquid, the right-hand side the external force. When the liquid is viscous, a resistance term has to be included on the right-hand side of (1.1). The resulting equation then provides a realistic description of the relative motion as long as a boundary layer on the bubble is either absent (no surface-active agents in the liquid) or does not separate from the bubble (when the presence of surface-active agents leads to a no-slip condition on the interface). Kelvin's result holds only when the fluid is

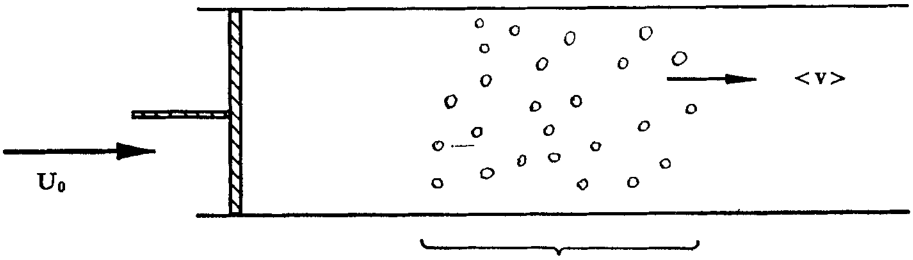

FIGURE 1. The liquid is accelerated from rest to a velocity $\mathbf{U}_0$ by the piston. As a result the bubbles in the cloud start to move with velocity $\langle \mathbf{v} \rangle$.

unbounded and is no longer applicable when other bodies are present in the fluid. The reason for this is that the external force on the test body includes forces due to the motion of these other bodies as well. In this paper we investigate the relative motion of spherical gas bubbles in a dilute gas-bubble/liquid mixture. More specifically, we shall calculate the average velocity $\langle \mathbf{v} \rangle$ of a cloud of identical spherical gas bubbles of radius a when the liquid in which the cloud finds itself is accelerated from rest to a velocity $\mathbf{U}_0$.

We shall also consider the momentum flux in the mixture. The reason for doing this is the following. As mentioned above, associated with the rectilinear motion of a single sphere through a fluid is an impulse $\frac{1}{2}\rho(v-u)\tau$. In a mixture containing n bubbles per unit volume, there is a flow of impulse, per unit surface normal to the flow, of magnitude $\frac{1}{2}\rho\tau n v(v-u)$. In describing two-phase flows, various writers have included the spatial rate of change of this term in the momentum equation, identifying momentum with impulse. Because this cannot be justified, it is desirable to know in what way, if any, the relative motion enters the expression for the momentum flux in the mixture. We consider the situation sketched in figure 1. A cloud of N spherical bubbles of radius a is dispersed randomly in a volume V in an infinite duct whose diameter is large compared with any length scale associated with the bubbles. At $t = 0$ the fluid far away from the cloud is instantaneously accelerated, e.g. by a piston, to a velocity $\mathbf{U}_0$. We want to know the average of the velocities which the bubbles assume. We define

$$n = N/V \tag{1.2}$$

as the number density and

$$\alpha = \tfrac{4}{3}\pi n a^3 \tag{1.3}$$

as the concentration by volume of bubbles. We assume that α is small with respect to unity. Our analysis to find the average bubble velocity runs parallel, in many respects, to that by Batchelor (1972) in calculating the sedimentation speed of a cloud of heavy particles in a liquid.

We need to know (i) the velocity which a single bubble assumes when an arbitrary velocity field is instantaneously generated in the ambient liquid and (ii) the velocity which a pair of bubbles acquires when the ambient liquid is impulsively accelerated. In the next two sections we consider these problems, because they have not been considered in the literature. Voinov, Voinov & Petrov (1973) investigated the hydrodynamic interaction between bodies in a perfect fluid. However, their analysis is restricted to bodies which are a large distance apart. In the present problem it will emerge that it is the interactions between bodies which are close to each other which are of primary importance.

2. The motion of a spherical bubble in an arbitrary potential flow

We consider a perfect fluid, at rest for times $t < 0$, in which a spherical gas bubble of radius a is immersed. At $t = 0$ a velocity field which has, in the absence of the bubble, a potential ϕ_0, is instantaneously generated in the fluid. As a result of the motion of the liquid the bubble will assume a velocity $\mathbf{v}$, and the resultant potential will be $\phi_0 + \phi_1$. Since the bubble can be regarded as massless, the resultant force exerted on the bubble by the liquid is zero at all times and therefore, with pressure p and surface element dA,

$$\int_{t=0_-}^{t=0_+} dt \int p\,\mathbf{dA} = 0, \tag{2.1}$$

where $t = 0_\mp$ indicate times just before and just after $t = 0$, respectively. By using Bernoulli's theorem it follows from (2.1) that

$$\int (\phi_0 + \phi_1)\,\mathbf{dA} = 0. \tag{2.2}$$

The meaning of (2.2) is that the impulsive forces on the sphere generated by the original motion of the liquid and by the relative motion between liquid and bubble are equal but opposite in sign. Apart from (2.2), the resulting potential has also to satisfy the boundary condition

$$\nabla(\phi_0 + \phi_1)\,.\,\mathbf{n} = \mathbf{v}\,.\,\mathbf{n} \tag{2.3}$$

on the sphere, $\mathbf{n}$ being a unit vector normal to the surface of the sphere. If $\mathbf{v}$ were known, (2.3) would, for given ϕ_0, uniquely determine ϕ_1. However, $\mathbf{v}$ is unknown and must be such as to satisfy (2.2). In solving this problem, two properties of the potential flow around a sphere are of great assistance. The first is that, for any harmonic function with singularities outside the sphere, the surface and volume averages over the sphere are equal to the value of that function at the centre of the sphere (see, for example, Kellog 1953, p. 223). Thus, if the value of $\nabla\phi_0$ at the centre of the sphere is denoted by $(\nabla\phi_0)_c$, we have

$$\int \phi_0\,\mathbf{dA} = \int \nabla\phi_0\,d\tau = (\nabla\phi_0)_c\,\tau, \tag{2.4}$$

τ being, as before, the volume of the sphere. The second useful property is given by Weiss's sphere theorem (Milne-Thomson 1968, p. 520). This states that, if in an unbounded potential flow with potential ϕ a sphere is placed at $\mathbf{r} = 0$ and if the singularities of ϕ are all outside the sphere, the resulting potential is

$$\phi + \frac{1}{a}\int_0^{a^2/r} R\,\frac{\partial}{\partial R}\{\phi(R, \omega, \theta)\}\,dR, \tag{2.5}$$

where (R, ω, θ) are spherical co-ordinates with origin at $\mathbf{r} = 0$. We cannot apply

Weiss's theorem to ϕ_0 because in the motion described by (2.5) the sphere is kept, by external forces, at rest, whereas our spherical bubble moves as a result of the impulsive forces associated with the generation of ϕ_0. We can apply Weiss's theorem, however, to that part of ϕ_0 which produces no motion at the centre of the sphere. Denoting $(\nabla\phi_0)_c$ by $\mathbf{u}$, we write ϕ_0 as

$$\phi_0 = \mathbf{u}\,.\,\mathbf{r} + \tilde{\phi}_0. \tag{2.6}$$

Because $\int (\mathbf{u}\,.\,\mathbf{r})\,\mathbf{dA} = \mathbf{u}\tau$, we have, on account of (2.4),

$$\int \tilde{\phi}_0 \mathbf{dA} = 0. \tag{2.7}$$

The response of the bubble to $\tilde{\phi}_0$ can be obtained from Weiss's theorem (2.5), whereas the response to the uniform flow with velocity $\mathbf{u}$ is well known: a doublet of strength $\frac{1}{2}a^3(\mathbf{v}-\mathbf{u})$ at the centre of the bubble. The potential $\phi_0+\phi_1$ therefore becomes

$$\phi_0+\phi_1 = \mathbf{u}\,.\,\mathbf{r} + \tilde{\phi}_0 - \frac{(\mathbf{v}-\mathbf{u})\,a^3\,.\,\mathbf{r}}{2r^3} + \frac{1}{a}\int_0^{a^2/r} R\frac{\partial}{\partial R}\tilde{\phi}_0(R,\omega,\theta)\,dR. \tag{2.8}$$

This potential, while satisfying (2.3), must satisfy (2.2) as well. At $r = a$, the last term on the right-hand side of (2.8) may be written as

$$\tilde{\phi}_0(a,\omega,\theta) - \frac{1}{a}\int_0^a \tilde{\phi}_0(R,\theta,\omega)\,dR.$$

On account of (2.7) and because $\tilde{\phi}_0$ is regular inside the sphere the integral of this expression over the surface of the sphere vanishes. We must choose $\mathbf{v}$ such that, upon substitution of (2.8) into (2.2), the integral comprising the first and third terms on the right-hand side of (2.8) is zero. As is easily verified, this is the case for

$$\mathbf{v} = 3\mathbf{u} = 3(\nabla\phi_0)_c. \tag{2.9}$$

We have in this way obtained the interesting result that a massless sphere moves, in an impulsively generated flow, with three times the velocity that this flow has at the location at $t = 0$ of the centre of the sphere. For a uniform flow, this follows immediately from (1.1). Note that the result expressed by (2.9) is independent of the radius of the sphere.

3. The motion of a pair of spheres in a perfect liquid

In this section we deal with a pair of identical bubbles immersed in a liquid which is instantaneously accelerated to a uniform velocity $\mathbf{U}_0$. We consider first the configuration of figure 2, where the line of centres is aligned with $\mathbf{U}_0$.

Let the unknown velocity of each of the spheres be $\mathbf{v}$. If sphere B were alone in the liquid the velocity distribution would be

$$\mathbf{u} = \mathbf{U}_0 - \frac{\mathbf{U}_0 a^3}{r^3} + \frac{3\mathbf{U}_0\,.\,a^3\mathbf{r}}{r^5}\mathbf{r}, \quad r > a, \tag{3.1}$$

$\mathbf{r}$ being measured with respect to the centre O_2 of B. This is the velocity field of

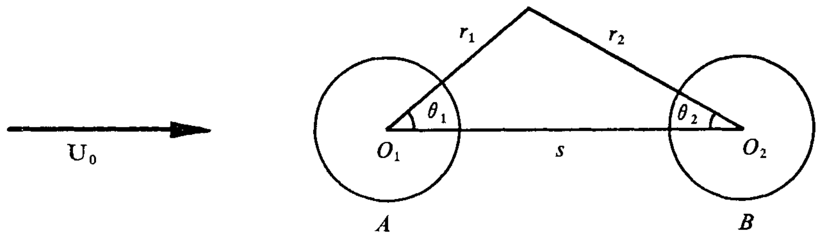

FIGURE 2. Two identical bubbles with their centres a distance s apart aligned with the liquid velocity $\mathbf{U}_0$.

a sphere moving with velocity $3\mathbf{U}_0$ through a liquid with velocity $\mathbf{U}_0$. The velocity at O_1 is

$$\mathbf{u}_1 = \mathbf{U}_0(1+2a^3/s^3), \tag{3.2}$$

where s is the distance between the centres. If sphere A were now placed in the liquid and only the boundary condition on A satisfied the sphere would, according to our previous result (2.9), move with velocity $3\mathbf{u}_1$. However, the boundary condition on B also has to be satisfied, which will give rise to an additional velocity $\mathbf{w}$ induced at the centre of A by its image in B, whence

$$\mathbf{v} = 3(\mathbf{u}_1+\mathbf{w}). \tag{3.3}$$

To find $\mathbf{w}$ we have to solve exactly the problem of finding the potential

$$\phi = \mathbf{U}_0 . \mathbf{r} + \psi \tag{3.4}$$

under the conditions

$$\nabla\psi . \mathbf{n} = (\mathbf{v}-\mathbf{U}_0) . \mathbf{n} \tag{3.5}$$

on each of the spheres A and B and [cf. (2.2)]

$$\int_A \phi \, \mathrm{d}\mathbf{A} = 0. \tag{3.6}$$

The potential of the motion of two spheres in the direction of the line of centres is treated in Lamb (1932, §98) by using the method of reflexions. The potential is found as an infinite series but the computation of the coefficients is a matter of great algebraic complexity. Recently Jeffrey (1973) used for two-sphere potential problems another method which he calls the method of twin spherical expansions.

Dr Jeffrey pointed out to the author that the results of Jeffrey (1973) could be used for the present problem and provided the author with the necessary information not in Jeffrey (1973). In the appendix, by Dr Jeffrey, the most important information is given. Jeffrey (1973) considers two spheres of heat conductivity λ_2 in a material of conductivity λ_1. In the absence of the spheres the temperature field is $T = \mathbf{G}.\mathbf{r}$. In the presence of the spheres the temperature field is written as

$$T = \mathbf{G}.\mathbf{r}+\psi. \tag{3.7}$$

For non-conducting spheres the boundary condition is

$$\nabla\psi . \mathbf{n} = -\mathbf{G}.\mathbf{n} \tag{3.8}$$

on each of the spheres. Upon introducing

$$\phi_1 = -(\mathbf{v}-\mathbf{U}_0).\mathbf{r}+\psi, \tag{3.9}$$

with the condition (3.5) on ψ, it is seen that our problem is formally the same as this steady heat-conduction problem when we replace T by ϕ_1 and $\mathbf{G}$ by $-(\mathbf{v}-\mathbf{U}_0)$. For the case of two spheres aligned with $\mathbf{G}$, Jeffrey finds

$$\mathbf{i}.\int_A T\,\mathbf{dA} = \tau G L_1\left(\frac{a}{s}\right), \tag{3.10}$$

with

$$L_1\left(\frac{a}{s}\right) = \frac{3}{2}\sum_{p=0}^{\infty} K_{01p}\left(\frac{a}{s}\right)^p, \tag{3.11}$$

$\mathbf{i}$ being the unit vector in the direction of the line of centres and τ being the volume of a sphere, as before. The coefficients K_{01p} follow (see appendix) from

$$K_{mn0} = (-1)^m \delta_{1n}, \tag{3.12}$$

$$K_{mnp} = (-1)^{m-1}\frac{n}{n+1}\sum_{q-1}^{\frac{1}{2}(p-n-3)}\binom{n+q}{n+m} K_{mq\,(p-n-q-1)}, \quad p \geqslant 1, \tag{3.13}$$

$$\delta_{1n} = \begin{Bmatrix} 0 & \text{for} & n = 1, \\ 1 & \text{for} & n = 1. \end{Bmatrix} \tag{3.14}$$

From (3.4), (3.6) and (3.9) it follows that the condition on ϕ_1 is

$$\mathbf{i}.\int_A \phi_1 \mathbf{dA} = -(v-U_0)\tau - U_0\tau. \tag{3.15}$$

With $\phi_1 = T$ and $\mathbf{G} = -(\mathbf{v}-\mathbf{U}_0)$ we deduce from (3.10) and (3.15) that

$$(v-U_0)+U_0 = (v-U_0)\,L_1(a/s). \tag{3.16}$$

Making use of $K_{010} = 1$ and $K_{011} = K_{012} = 0$ it follows that

$$\tfrac{1}{2}(v-U_0)\left\{1+3\sum_{p=3}^{\infty} K_{01p}\left(\frac{a}{s}\right)^p\right\} = U_0. \tag{3.17}$$

The interaction between the two spheres may be expressed in terms of the virtual mass. We write, in the spirit of (1.1),

$$\frac{d}{dt}m(\mathbf{v}-\mathbf{U}_0) = \frac{d}{dt}\rho\tau\mathbf{U}_0.$$

Then the virtual mass is given by the quantity in braces in (3.17) multiplied by $\frac{1}{2}\rho\tau$. Calculating a few coefficients K_{01p}, we obtain

$$m = \tfrac{1}{2}\rho\tau\left\{1-3\left(\frac{a}{s}\right)^3+3\left(\frac{a}{s}\right)^6+9\left(\frac{a}{s}\right)^8-3\left(\frac{a}{s}\right)^9+\ldots\right\}. \tag{3.18}$$

The added mass is smaller than for a single bubble, for which $a/s = 0$. The physical explanation is that each bubble induces [see (3.2)] at the centre of the other a velocity which is in the direction of $\mathbf{U}_0$. The accelerated liquid can therefore bring the pair of bubbles to a larger velocity than it could a single bubble. From (3.17) it follows that

$$\mathbf{v} = 3\mathbf{U}_0\left\{1+\sum_{p=3}^{\infty} K_{01p}\left(\frac{a}{s}\right)^p\right\}\bigg/\left\{1+3\sum_{p=3}^{\infty} K_{01p}\left(\frac{a}{s}\right)^p\right\}. \tag{3.19}$$

The velocity $\mathbf{w}$ induced in the centre of A by its image in B is given, using (3.2), (3.3), (3.19) and $K_{013} = -1$, by

$$\mathbf{w} = 2\mathbf{U}_0 F_1(a/s), \tag{3.20}$$

where

$$F_1\left(\frac{a}{s}\right) = -\frac{\sum_{p=6}^{\infty}\{K_{01p}+3K_{01(p-3)}\}\left(\frac{a}{s}\right)^p}{1+3\sum_{p=3}^{\infty} K_{01p}\left(\frac{a}{s}\right)^p}. \tag{3.21}$$

Next we consider the situation of figure 3, where the line connecting O_1 to O_2 is at right angles to $\mathbf{U}_0$. In this case the velocity at O_1 in the presence of sphere B but without sphere A is

$$\mathbf{u}_1 = \mathbf{U}_0(1-a^3/s^3), \tag{3.22}$$

the induced velocity being directed against $\mathbf{U}_0$. The corresponding configuration in the steady heat-conduction problem considered by Jeffrey leads here to

$$\mathbf{j}\cdot\int T\,\mathrm{d}\mathbf{A} = \tau G L_2(a/s), \tag{3.23}$$

where $\mathbf{j}$ is a unit vector in the direction normal to the line of centres and

$$L_2\left(\frac{a}{s}\right) = -\frac{3}{2}\sum_{p=0}^{\infty} K_{11p}\left(\frac{a}{s}\right)^p. \tag{3.24}$$

In the same way as for the aligned case we can deduce from this that in our two-sphere situation

$$\tfrac{1}{2}(\mathbf{v}-\mathbf{U}_0)\left\{1-3\sum_{p=2}^{\infty} K_{11p}\left(\frac{a}{s}\right)^p\right\} = \mathbf{U}_0. \tag{3.25}$$

The added mass in the situation of figure 3 is therefore

$$m = \tfrac{1}{2}\rho\tau\left\{1-3\sum_{p=3}^{\infty} K_{11p}\left(\frac{a}{s}\right)^p\right\}, \tag{3.26}$$

or, upon calculation of a few coefficients,

$$m = \tfrac{1}{2}\rho\tau\left\{1+\frac{3}{2}\left(\frac{a}{s}\right)^3+\frac{3}{4}\left(\frac{a}{s}\right)^6+3\left(\frac{a}{s}\right)^8+\frac{3}{4}\left(\frac{a}{s}\right)^9+\dots\right\}. \tag{3.27}$$

In this case the virtual mass of each sphere in the pair is larger than that of a single sphere because the velocity induced in the centre of one sphere by the other sphere is, as follows from (3.22), opposed to $\mathbf{U}_0$. From (3.25) we have in the configuration of figure 3

$$\mathbf{v} = 3\mathbf{U}_0\frac{1-\sum_{p=3}^{\infty} K_{11p}\left(\frac{a}{s}\right)^p}{1-3\sum_{p=3}^{\infty} K_{11p}\left(\frac{a}{s}\right)^p}, \tag{3.28}$$

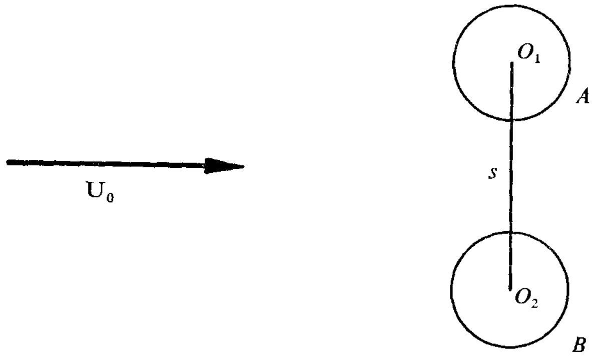

FIGURE 3. Two identical bubbles with their line of centres normal to the flow.

and hence, using (3.3) and (3.22),

$$\mathbf{w} = \mathbf{U}_0 F_2(a/s), \tag{3.29}$$

where

$$F_2\left(\frac{a}{s}\right) = \frac{\sum\limits_{p=6}^{\infty}\{2K_{11p} - 3K_{11(p-3)}\}\left(\dfrac{a}{s}\right)^p}{1 - 3\sum\limits_{p=3}^{\infty} K_{11p}\left(\dfrac{a}{s}\right)^p}. \tag{3.30}$$

4. Calculation of $\langle \mathbf{v} \rangle$

With the results of §§2 and 3 at hand we are now in a position to calculate the average velocity $\langle \mathbf{v} \rangle$ which the bubbles in the cloud of figure 1 attain upon acceleration of the liquid to a velocity $\mathbf{U}_0$. We borrow some of the notation in Batchelor (1972) and introduce $P(C_N)$ as the probability distribution of a configuration of N bubbles and $P(C_N|\mathbf{r}_0)$ as the conditional probability distribution, i.e. the probability distribution if there is a bubble at $\mathbf{r}_0$. The quantity $\langle \mathbf{v} \rangle$ is the average of the velocity $\mathbf{v}(\mathbf{r}_0, C_N)$ over all possible configurations of the N bubbles in the volume V. When $P(C_N)$ and $P(C_N \mid \mathbf{r}_0)$ are normalized such that

$$\frac{1}{N!}\int P(C_N)\,dC_N = \frac{1}{N!}\int P(C_N \mid \mathbf{r}_0)\,dC_N = 1, \tag{4.1}$$

$\langle \mathbf{v} \rangle$ is given by

$$\langle \mathbf{v} \rangle = \frac{1}{N!}\int \mathbf{v}(\mathbf{r}_0, C_N)\,P(C_N \mid \mathbf{r}_0)\,dC_N. \tag{4.2}$$

If we take the test bubble at $\mathbf{r}_0$ out of the mixture and consider the velocity $\mathbf{u}(\mathbf{r}_0, C_N)$ in the mixture, we know from the result (2.9) that, upon being placed in the liquid, the test bubble would move with velocity $3\mathbf{u}(\mathbf{r}_0, C_N)$ if only the boundary condition on the test bubble itself were to be satisfied. The boundary conditions on all other bubbles also have to be satisfied however. The images of the test bubble in the other bubbles induce an additional velocity $\mathbf{w}$ at $\mathbf{r}_0$, whence

$$\begin{aligned}\mathbf{v}(\mathbf{r}_0, C_N) &= 3\{\mathbf{u}(\mathbf{r}_0, C_N) + \mathbf{w}(\mathbf{r}_0, C_N)\} \\ &= 3\mathbf{U}_0 + 3\{\mathbf{u}(\mathbf{r}_0, C_N) - \mathbf{U}_0\} + 3\mathbf{w}(\mathbf{r}_0, C_N).\end{aligned} \tag{4.3}$$

Inserting this into (4.2) and using (4.1) gives

$$\langle \mathbf{v} \rangle = 3\mathbf{U}_0 + \frac{3}{N!}\int [\{\mathbf{u}(\mathbf{r}_0, C_N) - \mathbf{U}_0\} + \mathbf{w}(\mathbf{r}_0, C_N)]\, P(C_N \mid \mathbf{r}_0)\, dC_N. \tag{4.4}$$

To obtain a first approximation to $\langle \mathbf{v} \rangle$ in terms of the small quantity α, it is sufficient to (Batchelor 1972) consider the interaction of the test bubble with only one other bubble in each configuration, provided the relevant integrands fall off with distance $\mathbf{r}$ from $\mathbf{r}_0$ more rapidly than r^{-3}. The velocity $\mathbf{u}$ at $\mathbf{r}_0$ as a result of a bubble at $\mathbf{r}_0 + \mathbf{r}$ is given by (3.1) and from that relation it follows that this velocity minus $\mathbf{U}_0$, $\mathbf{u}(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{r}) - \mathbf{U}_0$, behaves at large distance like r^{-3}, making the reduction to interaction of the test bubble with just one other bubble impossible, because this leads to an integral which is not absolutely convergent. On the other hand the velocity $\mathbf{w}(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{r})$ induced at the centre $\mathbf{r}_0$ of the test bubble by its image in one other bubble with centre at $\mathbf{r}_0 + \mathbf{r}$ behaves [from (3.20), (3.21), (3.29) and (3.30)] like r^{-6}, so that with regard to $\mathbf{w}$ reduction to interaction with one other bubble is justified. To overcome the difficulty with the not absolutely convergent integral we use the technique due to Batchelor (1972; see also Batchelor 1974). In the present case this consists of observing that, since the volume flow is $\mathbf{U}_0$, we have, using (4.1),

$$\frac{1}{N!}\int \{\mathbf{u}(\mathbf{r}_0, C_N) - \mathbf{U}_0\}\, P(C_N)\, dC_N = 0, \tag{4.5}$$

where it should be recalled that $\mathbf{u}(\mathbf{r}_0, C_N)$ is the velocity in the suspension at $\mathbf{r}_0$. Note that $\mathbf{u}(\mathbf{r}_0, C_N)$ is multiplied in (4.5) by the ordinary probability distribution and in (4.4) by the conditional probability distribution. The point of Batchelor's technique now is to subtract (4.5) from (4.4) to obtain

$$\langle \mathbf{v} \rangle = 3\mathbf{U}_0 + \frac{3}{N!}\int \{\mathbf{u}(\mathbf{r}_0, C_N) - \mathbf{U}_0\}\{P(C_N \mid \mathbf{r}_0) - P(C_N)\}\, dC_N + \frac{3}{N!}\int \mathbf{w}(\mathbf{r}_0, C_N)\, P(C_N \mid \mathbf{r}_0)\, dC_N. \tag{4.6}$$

In the absence of any long-range order in the mixture we have for large r $P(C_N \mid \mathbf{r}_0) = P(C_N)$, and this removes the difficulty with the integral because now $P(C_N \mid \mathbf{r}_0) - P(C_N)$ tends to zero for large r. Both integrals in (4.6) may now be reduced to two-bubble interactions and we have to specify $P(\mathbf{r}_0 + \mathbf{r})$, the probability of finding a bubble centre at $\mathbf{r}_0 + \mathbf{r}$, and $P(\mathbf{r}_0 + \mathbf{r} \mid \mathbf{r}_0)$, the probability of finding a bubble centre at $\mathbf{r}_0 + \mathbf{r}$ given that there is one at $\mathbf{r}_0$. We shall assume that for $t < 0$ the bubbles are completely randomly distributed in the volume V. Since the distance between the two bubbles in a pair of bubbles remains unchanged as a result of the acceleration, this distribution is preserved at $t = 0_+$. Note that this is not so when we deal with bubbles of different sizes because then there is a relative velocity in each pair of bubbles which produces changes in the probability distribution. In that case it is necessary to investigate whether the system of N bubbles evolves towards a new steady probability distribution.

Clouds with bubbles of different sizes will not be considered here. For a completely random distribution

$$\left.\begin{aligned} P(\mathbf{r}_0+\mathbf{r}\mid\mathbf{r}_0) &= \begin{cases} 0 & \text{for } r < 2a, \\ n & \text{for } r \geqslant a, \end{cases} \\ P(\mathbf{r}_0+\mathbf{r}) &= n \quad \text{for all } r. \end{aligned}\right\} \tag{4.7}$$

Reducing now the right-hand side of (4.6) to two-bubble interactions, the first integral in the right-hand side becomes

$$\begin{aligned} \mathbf{I}_1 &= 3\int\{\mathbf{u}(\mathbf{r}_0,\mathbf{r}_0+\mathbf{r})-\mathbf{U}_0\}\{P(\mathbf{r}_0+\mathbf{r}\mid\mathbf{r}_0)-P(\mathbf{r}_0+\mathbf{r})\}\,d\mathbf{r} \\ &= -3n\int_{r<2a}\{\mathbf{u}(\mathbf{r}_0,\mathbf{r}_0+\mathbf{r})-\mathbf{U}_0\}\,d\mathbf{r}. \end{aligned} \tag{4.8}$$

To carry out the remaining integration we interchange the roles of the bubbles and consider $\mathbf{u}(\mathbf{r}_0+\mathbf{r},\mathbf{r}_0)$, i.e. the velocity at $\mathbf{r}_0+\mathbf{r}$ when there is a bubble at $\mathbf{r}_0$. For $r \leqslant a$, $\mathbf{u}(\mathbf{r}_0+\mathbf{r},\mathbf{r}_0)-\mathbf{U}_0$ is $2\mathbf{U}_0$, whereas in the region $a < r < 2a$, $\mathbf{u}$ is given by (3.1). Evaluating the integral in (4.8) using these expressions gives

$$\mathbf{I}_1 = -3n\int_{r\leqslant a}2\mathbf{U}_0\,d\mathbf{r}-3n\int_{a<r\leqslant 2a}\left(-\frac{\mathbf{U}_0a^3}{r^3}+\frac{3\mathbf{U}_0\cdot a^3\mathbf{r}}{r^5}\mathbf{r}\right)d\mathbf{r}.$$

The first term equals $-6\alpha\mathbf{U}_0$ while the second integral is zero, whence

$$\mathbf{I}_1 = -6\alpha\mathbf{U}_0. \tag{4.9}$$

The second integral in the right-hand side of (4.6) is, after reduction to interaction between two bubbles,

$$\mathbf{I}_2 = 3\int\mathbf{w}(\mathbf{r}_0,\mathbf{r}_0+\mathbf{r})\,P(\mathbf{r}_0+\mathbf{r}\mid\mathbf{r}_0)\,d\mathbf{r} = 3n\int_{r>2a}\mathbf{w}(\mathbf{r}_0+\mathbf{r},\mathbf{r}_0)\,d\mathbf{r}.$$

The velocity $\mathbf{w}(\mathbf{r}_0+\mathbf{r},\mathbf{r}_0)$ was calculated in §3 for the two independent configurations which a pair of bubbles may have with respect to $\mathbf{U}_0$. In order to find $\mathbf{w}$ for an arbitrary orientation of $\mathbf{r}$ with respect to $\mathbf{U}_0$ we write $\mathbf{U}_0$ as

$$\mathbf{U}_0 = \frac{\mathbf{U}_0\cdot\mathbf{r}}{r^2}\mathbf{r}+\left(\mathbf{U}_0-\frac{\mathbf{U}_0\cdot\mathbf{r}}{r^2}\mathbf{r}\right).$$

When the line joining the centres is parallel to $\mathbf{U}_0$, $\mathbf{w}$ is given by (3.20) and (3.21), while when it is perpendicular to $\mathbf{U}_0$, $\mathbf{w}$ is given by (3.29) and (3.30). We have for an arbitrary configuration, therefore,

$$\mathbf{w}(\mathbf{r}_0+\mathbf{r},\mathbf{r}_0) = \frac{\{2\mathbf{U}_0\cdot\mathbf{r}\}F_1(a/r)}{r^2}\mathbf{r}+\frac{\{\mathbf{U}_0r^2-(\mathbf{U}_0\cdot\mathbf{r})\,\mathbf{r}\}}{r^2}F_2(a/r).$$

We integrate first over the surface of a sphere with radius r. Using spherical polar co-ordinates, we get

$$\mathbf{w}' = \int_0^\pi 2\pi r^2\mathbf{w}\sin\theta\,d\theta = \tfrac{8}{3}\pi r^2\{F_1(a/r)+F_2(a/r)\}\,\mathbf{U}_0.$$

The integration over r from $2a$ to ∞ has to be carried out numerically using (3.21) for $F_1(a/r)$ and (3.30) for $F_2(a/r)$. The result is

$$\mathbf{I}_2 = 3n\int_{2a}^{\infty} \mathbf{w}'\,\mathbf{dr} = 6\alpha \mathbf{U}_0 \int_0^{\frac{1}{2}} z^{-4}\{F_1(z)+F_2(z)\}\,dz.$$

This integral was computed by the author's associate G. J. de Bruin. He carried out the integration with the upper bound in the infinite series in (3.21) and (3.30) ranging from $p = 6$ to $p = 40$, and obtained for the value of the integral 0·0744, correct to the fourth decimal place. Hence

$$\mathbf{I}_2 = 0{\cdot}4464\alpha \mathbf{U}_0. \tag{4.10}$$

Taking together the results (4.9) and (4.20) we obtain for $\langle \mathbf{v} \rangle$, which is the sum of $3\mathbf{U}_0$, $\mathbf{I}_1$ and $\mathbf{I}_2$,

$$\langle \mathbf{v} \rangle = 3\mathbf{U}_0(1-2\alpha+0{\cdot}15\alpha) = 3\mathbf{U}_0(1-1{\cdot}85\alpha), \tag{4.11}$$

correct to two decimal places. This result can be interpreted as follows. In the absence of bubbles the velocity of the liquid is $\mathbf{U}_0$. If one bubble were present in the liquid this would attain the velocity $3\mathbf{U}_0$, on the basis of (2.9). Hence to the zeroth approximation in α each bubble moves with velocity $3\mathbf{U}_0$. The resulting volume flow of bubbles is $3\mathbf{U}_0\alpha$. Since the volume occupied by the liquid is $1-\alpha$ the average liquid velocity is $\mathbf{U}_0(1-2\alpha)$ to maintain a volume flow of $\mathbf{U}_0$. If (2.9) held for a test bubble the average bubble velocity would be $3\mathbf{U}_0(1-2\alpha)$, which is $3\mathbf{U}_0+\mathbf{I}_1$. However, the average bubble velocity is not quite three times the average liquid velocity. The difference is the average of $3\mathbf{w}$, which is given by $\mathbf{I}_2$.

The result (4.14) may also be converted to an effective virtual mass. We write for a bubble in a suspension of concentration α

$$m = \tfrac{1}{2}\rho\tau(1+k\alpha)+O(\alpha^2).$$

Denoting $\langle \mathbf{v} \rangle$ by $\mathbf{V}$ we write in analogy with (1.1) and replacing $\frac{1}{2}\rho\tau$ by m,

$$d[m(\mathbf{V}-\mathbf{U}_0)]/dt = \rho\tau\, d\mathbf{U}_0/dt, \tag{4.12}$$

which gives
$$\mathbf{V} = 3\mathbf{U}_0(1-\tfrac{2}{3}k\alpha)+O(\alpha^2).$$

Equating the right-hand side to that of (4.11) gives $k = 2{\cdot}78\alpha$ and hence

$$m = \tfrac{1}{2}\rho\tau(1+2{\cdot}78\alpha)+O(\alpha^2). \tag{4.13}$$

An estimate of the effect on m of α was made by Zuber (1964). He used the result of the following problem. A sphere of radius a is located in the centre of another sphere with radius $b > a$, filled with liquid. The latter is accelerated from rest to a velocity U. What is the velocity attained by the sphere with radius a? This problem is dealt with in many textbooks, e.g. Milne-Thomson (1968, p. 523). Zuber (1964) took for the radius b

$$b = (3/4\pi n)^{\frac{1}{3}},$$

i.e. the average distance between spheres in a mixture with number density of spheres n. Then he obtained to order α^2

$$m = \tfrac{1}{2}\rho\tau(1+3\alpha) \tag{4.14}$$

as an estimate for m in a mixture of void fraction α. At first sight it seems surprising that this crude simplification of the real situation provides an estimate which, as follows from comparison with (4.13), is not far from the correct answer. On closer examination it becomes clear why this is the case. The term proportional to α in (4.13) stems from the displacement, the integral $\mathbf{I}_1$, and from the contribution of $\mathbf{w}$, the integral $\mathbf{I}_2$. They have a ratio of about 13. The first effect has to do with conservation of volume and is independent of the particular geometry of the boundaries of the flow. By his particular choice of b, Zuber (1964) obtains this contribution correctly. The contribution of $\mathbf{w}$ vanishes in his configuration because of symmetry. The symmetry is distorted when the sphere with radius a is not in the centre of the liquid-filled sphere. So the contribution of $\mathbf{I}_2$ is absent in Zuber's estimate, but since its magnitude is small compared with that of $\mathbf{I}_1$, his result (4.14) is close to the correct one.

The result (4.13) was calculated for the situation of figure 1, in which there is a cloud of bubbles immersed in pure liquid. In practice one is often concerned with a homogeneous mixture with uniformly dispersed bubbles. Such a mixture has, for small α, a density $\rho(1-\alpha)$ and an equation of motion

$$\rho(1-\alpha)\,d\mathbf{U}_l/dt = -\nabla p, \tag{4.15}$$

where $\mathbf{U}_l$ denotes the liquid velocity. Let the average bubble velocity be $\mathbf{V}$. The volume flow is

$$(1-\alpha)\,\mathbf{U}_l+\alpha\mathbf{V}. \tag{4.16}$$

When the mixture is accelerated, for example by the passage of a pressure wave, the relative velocity may, in the absence of viscosity, be calculated from

$$d[m(\mathbf{V}-\mathbf{U}_l)]/dt = \rho\,(1-\alpha)\,\tau\, d\mathbf{U}_l/dt, \tag{4.17}$$

where m is given by (4.13).

5. Calculation of momentum flux; comparison with experimental observation

The result (4.13) suggests an increase with α of the apparent virtual mass and it would be interesting to know if something like this has been observed in practice. The author is not aware of measurements which would enable a direct check on the relation (4.13). However Prins (1974) reported a series of interesting experimental results which have a bearing on the present work. In describing theoretically the momentum flux in a liquid/bubble mixture Prins included a term $mnV(V-U_l)$, where the symbols are those used in the preceding section. This amounts to counting the flux of impulse associated with the motion of the bubbles as momentum of the mixture. With

$$mn = \rho\alpha B(\alpha), \tag{5.1}$$

Prins (1974) wrote for the momentum equation of the steady flow of a bubbly liquid through a converging tube with cross-section $A(x)$

$$\rho\frac{d}{dx}\{(1-\alpha)\,AU_l^2\}+\rho\frac{d}{dx}\{B\alpha A\,V(V-U_l)\} = -A\,\frac{dp}{dx}-\sigma_w O-A(1-\alpha)\,\rho g. \tag{5.2}$$

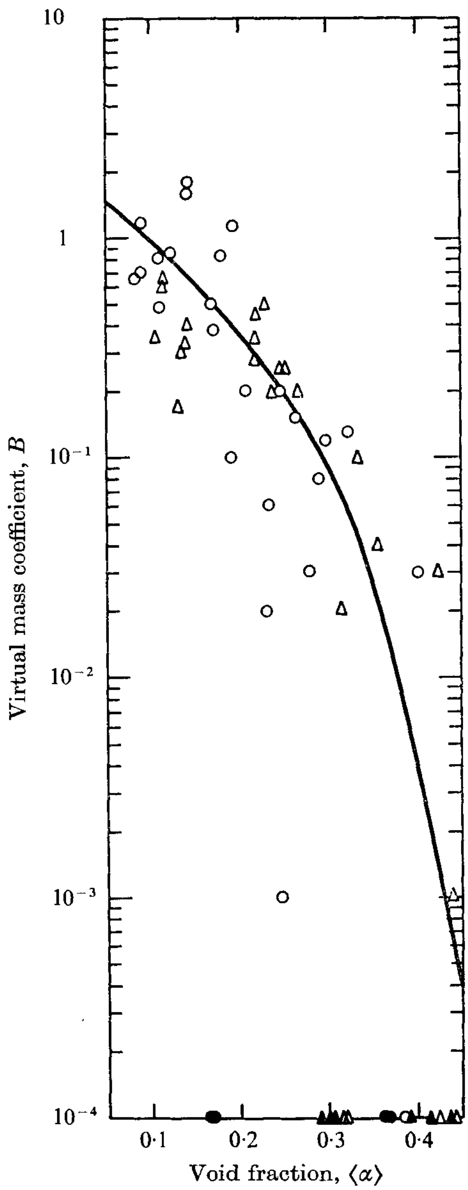

FIGURE 4. B as a function of α, from Prins (1974). Triangles and circles refer to experiments in different test facilities.

Here σ_w is the wall shear stress, O the circumference of the cross-section A, g the acceleration due to gravity and p the pressure in the mixture.

Next he carried out experiments in which $p(x)$ and $\alpha(x)$ were measured along the converging section of the tube. As a theoretical model, (5.2) together with other conservation equations and an equation like (1.1) were used, allowance being made in the latter for a friction force with drag coefficient C_D. Using independently obtained values for σ_w and C_D, numerical solution of these equations, combined with the measured values of $p(x)$ and $\alpha(x)$, permitted determination of B as a function of α (for details the reader is referred to the original report). From (5.1) it follows that $B(\alpha) = m/\rho\tau$, so that according to our results (4.19)

$$B(\alpha) = \tfrac{1}{2}(1+2{\cdot}78\alpha). \tag{5.3}$$

Figure 4, taken from Prins (1974), shows the results he obtained in the manner described above. There is a striking discrepancy between the qualitative behaviour of B in this figure, where it rapidly decreases with α, and our result (5.3), which predicts an increase of the virtual mass with α. Of course, in a real mixture the bubbles are not all of the same size nor are they perfectly spherical, but it becomes obvious from inspecting the various contributions to $B(\alpha)$ in (5.3) that the analysis would always show an increase of m with α, the numerical value 2·78 in (5.3) presumably taking a different value for a real mixture. This casts doubts on the validity of the second term on the left-hand side of (5.2), the relative-motion term. To settle this we calculate the flux of momentum in the mixture for a given α, a given gas velocity V and a given liquid velocity U_l. The volume flow is denoted by U_0, as before:

$$(1-\alpha)\,U_l+\alpha V = U_0. \tag{5.4}$$

Since we may, for small α, neglect the contribution of the gas to the momentum flux, we want to know the momentum flux in the liquid only. To this end we introduce a function $F(\mathbf{r}_0, C_N)$ which is unity when $\mathbf{r}_0$ is in the liquid and zero when $\mathbf{r}_0$ is in a bubble. Such a function has been used before in the analysis of suspensions (see, for example, Lundgren 1972) and is defined as

$$F(\mathbf{r}_0, C_N) = 1-\sum_{k=1}^{N} H(a-|\mathbf{r}_0-\mathbf{r}_k|\}, \tag{5.5}$$

where H is Heaviside's unit step function. With this definition of F

$$\frac{1}{N!}\int F(\mathbf{r}_0, C_N)\,P(C_N)\,dC_N = 1-\alpha. \tag{5.6}$$

The momentum flux in the liquid is given by

$$\mathbf{M} = \frac{\rho}{N!}\int F(\mathbf{r}_0, C_N)\,\mathbf{u}(\mathbf{r}_0, C_N)\,\mathbf{u}(\mathbf{r}_0, C_N)\,P(C_N)\,dC_N. \tag{5.7}$$

Upon multiplication of the dyadic $\mathbf{M}$ in (5.7) by the unit vector $\mathbf{i}$ in the direction of $\mathbf{U}_0$, we obtain the average momentum flux in that direction. The contribution of $(\mathbf{u}\,.\,\mathbf{i})^2$ to the right-hand side of (5.7) is the only one which survives after averaging, whence

$$M = \frac{\rho}{N!}\int F(\mathbf{r}_0, C_N)\,(\mathbf{u}\,.\,\mathbf{i})^2\,P(C_N)\,dC_N. \tag{5.8}$$

We write $\mathbf{u}(\mathbf{r}_0, C_N)$, the local velocity in the liquid or in a bubble, as

$$\mathbf{u} = \mathbf{U}_0+\mathbf{u}',$$

and insert this into (5.8). Because

$$\frac{1}{N!}\int F(\mathbf{r}_0, C_N)\,\mathbf{u}'(r_0, C_N)\,P(C_N)\,dC_N = (1-\alpha)\,(U_l-U_0) = -\alpha(V-U_0)$$

and because of (5.6) we obtain

$$M = U_0^2(1-\alpha) - 2\alpha U_0(V-U_0) + \frac{1}{N!}\int F(\mathbf{r}_0, C_N)\{\mathbf{u}'.\mathbf{i}\}^2 P(C_N)\, dC_N. \tag{5.9}$$

The integral in (5.9) can be reduced to averaging over one bubble in each configuration. With a bubble of velocity $\mathbf{V}$, and when there is only one bubble in each configuration, $\mathbf{u}'$ is given by

$$\mathbf{u}' = -\frac{(\mathbf{V}-\mathbf{U}_0)\,a^3}{2r^3} + 3\frac{(\mathbf{V}-\mathbf{U}_0)\,a^3.\mathbf{r}}{2r^5}\mathbf{r},$$

or in spherical polar co-ordinates

$$\mathbf{u}'.\mathbf{i} = -\frac{(V-U_0)}{2}\left(\frac{a^3}{r^3} - \frac{3a^2\cos^2\theta}{r^3}\right), \quad r > a, \tag{5.10}$$

The square $(\mathbf{u}'.\mathbf{i})^2$ therefore decreases as r^{-6} and this is rapidly enough to permit averaging over one bubble in each configuration. The integral in (5.9) therefore reduces, using (5.5), to

$$\int_{0<r<\infty}\{\mathbf{u}'(\mathbf{r}_0, \mathbf{r}_0+\mathbf{r}).\mathbf{i}\}^2 P(\mathbf{r}_0+\mathbf{r})\,\mathbf{dr} - \int_{0<r<\infty}\{\mathbf{u}'(\mathbf{r}_0, \mathbf{r}_0+\mathbf{r}).\mathbf{i}\}^2 H(a-r)\,P(\mathbf{r}_0+\mathbf{r})\,\mathbf{dr} = n\int_{r>a}\{\mathbf{u}'(\mathbf{r}_0, \mathbf{r}_0+\mathbf{r}).\mathbf{i}\}^2\,\mathbf{dr}.$$

Inserting (5.10), we obtain

$$\frac{n(V-U_0)^2}{4}\int_a^\infty dr\int_0^\pi 2\pi r^2\left\{\left(\frac{a}{r}\right)^3 - \frac{3a^3\cos^2\theta}{r^3}\right\}^2 \sin\theta\, d\theta = \tfrac{1}{5}\alpha(V-U_0)^2. \tag{5.11}$$

From this result we find

$$M/\rho = (1-\alpha)\,U_0^2 - 2\alpha U_0(V-U_0) + \tfrac{1}{5}\alpha(V-U_0)^2. \tag{5.12}$$

For practical purposes the liquid velocity is more convenient than U_0. From (5.4) and (5.12) we finally obtain, neglecting terms of order α^2,

$$M/\rho = U_l^2(1-\alpha) + \tfrac{1}{5}\alpha(V-U_l)^2. \tag{5.13}$$

Prins (1974) and others use the expression [cf. (5.2)]

$$M/\rho = U_l^2(1-\alpha) + \alpha B V(V-U_l). \tag{5.14}$$

The result (5.13) of our analysis suggests that the second term on the right-hand side of (5.14) should be proportional to $\alpha(V-U_1)^2$. If this were written as in (5.14), we should need for agreement

$$B(\alpha) \sim (V-U_l)/V. \tag{5.15}$$

If a mixture is accelerated from rest we may use (4.17) for the relative velocity. When in addition m increases with α, represented by

$$m = \tfrac{1}{2}\rho\tau(1+\kappa\alpha), \quad \kappa > 0,$$

then the expression (5.15) for $B(\alpha)$ behaves as

$$B(\alpha) \sim 1-\tfrac{1}{3}(1+\kappa)\alpha, \tag{5.16}$$

which shows a decrease in $B(\alpha)$ with α, as observed by Prins (figure 4). The constant κ in (5.16) should be much larger than $\kappa = 2{\cdot}78$, the value corresponding to (4.13), in order to obtain quantitative agreement with Prins' results. However a momentum flux $mnV(V-U_l)$ cannot be made to agree with Prins' results because we found that the effective virtual mass increases with α. A flux, as calculated here [see (5.13)], explains Prins' result qualitatively.

6. Conclusion

It is shown that a cloud of gas bubbles immersed in an accelerated liquid attains as a result of hydrodynamic interaction a velocity which is less than that which a single bubble would attain. This result may be interpreted as corresponding to an increase in the virtual mass. An expression for the momentum flux is derived by statistical averaging. This appears to differ from the one previously used in the literature. The expression obtained here, together with the result on virtual mass, explains qualitatively the observed behaviour of the contribution to the momentum flux of relative motion between bubbles and liquid.

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Appendix. The details of the solution of the two-sphere problem

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Equation numbers quoted from Jeffrey (1973) will be prefixed with J to distinguish them from those of the present paper. As pointed out in the main paper, the solution to the heat-conduction problem given in Jeffrey (1973) is needed only in the limit of non-conducting spheres, and so the parameters α and β_n defined in (J3.2) can be set equal to 0 and $-n/(n+1)$ and removed from the equations. If the applied field $\mathbf{G}$ is written $G_0\mathbf{i}+G_1\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors parallel and perpendicular to the line of centres, then the solution outside the spheres is (J5.2):

$$T = \mathbf{G}.\mathbf{x}+\sum_{m=0}^{1}\sum_{n=m}^{\infty} G_m\left\{g^{(1)}_{mn}\left(\frac{a}{r_1}\right)^{n+1} P^m_n(\cos\theta_1)+g^{(2)}_{mn}\left(\frac{a}{r_2}\right)^{n+1} P^m_n(\cos\theta_2)\cos m\phi,\right. \tag{A 1}$$

where the co-ordinates are given in figure 2 and the $g^{(1)}_{mn}$ and $g^{(2)}_{mn}$ are coefficients to be determined. Application of the boundary condition $\partial T/\partial n = 0$ gives

$$-\left(1+\frac{1}{n}\right)g^{(i)}_{mn}+\sum_{q=m}^{\infty}\binom{n+q}{n+m}g^{(3-i)}_{mq}\left(\frac{a}{s}\right)^{n+q+1} = (-1)^{i(m-1)}a\delta_{1n}, \tag{A 2}$$

where the R used in (J5.5) has been replaced by s for consistency with the main paper. Solving (A 2) for $g^{(i)}_{mn}$ is now the main difficulty and several methods for this have been proposed (Ross 1970). Here each coefficient is expressed as a series in a/s. Using symmetry to replace the $g^{(i)}_{mn}$ by

$$ag_{mn} = g^{(1)}_{mn} = (-1)^{m-1}g^{(2)}_{mn}, \tag{A 3}$$

one writes the solution as

$$g_{mn} = \frac{1}{2}\sum_{p=0}^{\infty}K_{mnp}\left(\frac{a}{s}\right)^{p}, \tag{A 4}$$

and calculates the coefficients K_{mnp} by substituting (A 4) into (A 2) and equating coefficients of powers of a/s. Thus (A 2) becomes

$$-\left(1+\frac{1}{n}\right)\frac{1}{2}\sum_{p=0}^{\infty}K_{mnp}\left(\frac{a}{s}\right)^{p}+\sum_{q=m}^{\infty}\binom{n+q}{n+m}\frac{1}{2}\sum_{p=0}^{\infty}(-1)^{m-1}K_{mqp}\left(\frac{a}{s}\right)^{n+p+q+1} = (-1)^{(m-1)}\delta_{1n},$$

giving

$$K_{mn0} = (-1)^{m}\delta_{1n} \tag{A 5}$$

and

$$K_{mnp} = (-1)^{(m-1)}\frac{n}{n+1}\sum_{q=1}^{\frac{1}{2}(p-n-3)}\binom{n+q}{n+m}K_{mq(p-q-n-1)}, \quad p \geqslant 1. \tag{A 6}$$

The calculation for sphere A of

$$\mathbf{S} = \int T\mathbf{n}\,dA = \mathbf{i}\int T\cos\theta_1\,dA+\mathbf{j}\int T\sin\theta_1\cos\phi\,dA$$

proceeds by using (J5.3) and (A 3) to rewrite (A 1) as

$$T = \mathbf{G}.\mathbf{x}+\sum_{m=0}^{1}\sum_{n=m}^{\infty}aG_m g_{mn}\left(\frac{a}{r_1}\right)^{n+1}P^m_n(\cos\theta_1)\cos m\phi$$
$$+\sum_{m=0}^{1}\sum_{n=m}^{\infty}(-1)^{m-1}aG_m g_{mn}\left(\frac{a}{s}\right)^{n+1}\sum_{q=m}^{\infty}\binom{n+q}{q+m}\left(\frac{r_1}{s}\right)^{q}P^m_q(\cos\theta_1)\cos m\phi.$$

This expression for T can now be substituted into the definition of $\mathbf{S}$ and the integration done simply by using the orthogonality of spherical harmonics. The result is

$$\mathbf{S} = \tau\mathbf{G}+\mathbf{i}\tau\left\{G_0g_{01}-\sum_{n=0}^{\infty}G_0g_{0n}\left(\frac{a}{s}\right)^{n+2}\binom{n+1}{1}\right\}$$
$$+\mathbf{j}\tau\left\{G_1g_{11}+\sum_{n=1}^{\infty}G_1g_{1n}\left(\frac{a}{s}\right)^{n+2}\binom{n+1}{2}\right\}.$$

The two summations can be removed using (A 2), the expression for $\mathbf{S}$ then becoming

$$\mathbf{S} = 3\tau G_0\mathbf{i}g_{01}-3\tau G_1\mathbf{j}g_{11}.$$

Finally, substituting the solution (A 4) gives the expression for **S** as

$$\mathbf{S} = \tau G_0 \mathbf{i} \frac{3}{2} \sum_{p=0}^{\infty} K_{01p} \left(\frac{a}{s}\right)^p - \tau G_1 \mathbf{j} \frac{3}{2} \sum_{p=0}^{\infty} K_{11p} \left(\frac{a}{s}\right)^p.$$

The results for **S** are quoted in the main paper as (3.10) and (3.23) and the recurrence relations (A 5) and (A 6) as (3.12) and (3.13). The method just described gives the value of **S** as a series in a/s, just as the method of reflexions does; the advantage of this method is that the recurrence relations (A 5) and (A 6) enable any number of terms in the series to be calculated rapidly, something not possible using reflexions.

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